UNIVERSITY OF CAMBRIDGE INTERNATIONAL EXAMINATIONS

GCE Advanced Level

MARK SCHEME for the May/June 2010 question paper

for the guidance of teachers

9231 FURTHER MATHEMATICS

9231/11 Paper 11, maximum raw mark 100

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes must be read in conjunction with the question papers and the report on the examination.

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UNIVERSITY of CAMBRIDGE International Examinations

Mark Scheme Notes

Marks are of the following three types:

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

- When a part of a question has two or more "method" steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly when there are several B marks allocated. The notation DM or DB (or dep*) is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

- The symbol √ implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only. A and B marks are not given for fortuitously "correct" answers or results obtained from incorrect working.

- Note: B2 or A2 means that the candidate can earn 2 or 0.
  B2/1/0 means that the candidate can earn anything from 0 to 2.

  The marks indicated in the scheme may not be subdivided. If there is genuine doubt whether a candidate has earned a mark, allow the candidate the benefit of the doubt. Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored.

- Wrong or missing units in an answer should not lead to the loss of a mark unless the scheme specifically indicates otherwise.

- For a numerical answer, allow the A or B mark if a value is obtained which is correct to 3 s.f., or which would be correct to 3 s.f. if rounded (1 d.p. in the case of an angle). As stated above, an A or B mark is not given if a correct numerical answer arises fortuitously from incorrect working. For Mechanics questions, allow A or B marks for correct answers which arise from taking *g* equal to 9.8 or 9.81 instead of 10.

The following abbreviations may be used in a mark scheme or used on the scripts:

AEF Any Equivalent Form (of answer is equally acceptable)

AG Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)

BOD Benefit of Doubt (allowed when the validity of a solution may not be absolutely clear)

CAO Correct Answer Only (emphasising that no "follow through" from a previous error is allowed)

CWO Correct Working Only – often written by a 'fortuitous' answer

ISW Ignore Subsequent Working

MR Misread

PA Premature Approximation (resulting in basically correct work that is insufficiently accurate)

SOS See Other Solution (the candidate makes a better attempt at the same question)

SR Special Ruling (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)

Penalties

MR –1 A penalty of MR –1 is deducted from A or B marks when the data of a question or part question are genuinely misread and the object and difficulty of the question remain unaltered. In this case all A and B marks then become "follow through √" marks. MR is not applied when the candidate misreads his own figures – this is regarded as an error in accuracy. An MR–2 penalty may be applied in particular cases if agreed at the coordination meeting.

PA –1 This is deducted from A or B marks in the case of premature approximation. The PA –1 penalty is usually discussed at the meeting.

1	$1+1+(y')^3 = 29 \Rightarrow (y')^3 = 27 \Rightarrow y' = 3$	B1
	$2x + 2yy'$, $+3(y')^2 y'' = 0$	M1A1, A1
	$2 - 6 + 27y'' = 0 \Rightarrow y' = \dfrac{4}{27}$	A1
		[5]

2	**(i)** Sketch of C:	
	Approximately correct shape and location for $0 \le \theta < 2\pi$	B1
	Shows initial line to be tangential to C at the pole	B1
	Asymptotic approach to circle $r = a$	B1
		[3]
	(ii) $A = (a^2/2)\int_{\ln 2}^{\ln 4}(1 - 2e^{-\theta} + e^{-2\theta})d\theta$	M1A1
	$= (a^2/2)[\theta + 2e^{-\theta} - (1/2)e^{-2\theta}]_{\ln 2}^{\ln 4}$	A1
	$= \ldots = (a^2/2)(\ln 2 - 13/32)$ (AG)	A1
		[4]

3	$\dfrac{ds}{dt} = \sqrt{t(t^2+4) + t^2(4-t)} = \sqrt{8t^2} = 2\sqrt{2}t$	B1
	$s = 2\sqrt{2}\int_0^2 t\,dt = \sqrt{2}\left[t^2\right]_0^2 = 4\sqrt{2}$ (AG)	M1A1
		[3]
	$y = (1/3)(4 - t^2)^{3/2}$	B1
	$S = +2\pi/3\int_0^2 2\sqrt{2}t(4-t^2)^{3/2}dt$	M1
	$= \ldots = \left[-(4\sqrt{2}\pi/15)(4-t^2)^{5/2}\right]_0^2$	A1
	$= 128\sqrt{2}\pi/15$	A1
		[4]

4	$(N + 1/2)^6 - 1/64 = 6S_N + (5/4)N^2(N+1)^2 + 3N(N+1)/16$	M1A1A1
	M1 for application of difference method:	
	A1 for LHS correct: A1 for RHS correct	
	$S_N = (1/6)(N + 1/2)^6 - (5/24)N^2(N+1)^2 - (1/32)N(N+1) - 1/384$	
	Or $\dfrac{1}{6}\left\{(N+\dfrac{1}{2})^6 - (\dfrac{1}{2})^6 - \dfrac{5N^2(N+1)^2}{4} - \dfrac{3}{16}N(N+1)\right\}$	A1
		[4]
	(i) For $\lambda = 6$, $S_\infty = 1/6$	B2
	(ii) For $\lambda > 6$, $S_\infty = 0$	B1
		[3]

5 $D\,[(x^2/2)(\ln x)^n] = x(\ln x)^n + (nx/2)(\ln x)^{n-1}$ M1

$\Rightarrow [(x^2/2)(\ln x)^n]_1^e = I_n + (n/2)I_{n-1}$ A1

$\Rightarrow \ldots \Rightarrow I_n = e^2/2 - (n/2)I_{n-1}$ for $n \geqslant 2$

$\Rightarrow I_{n+1} = e^2/2 - (n+1)2I_{n-1}$ for $n \geqslant 1$ A1

[3]

OR

$\int_1^e x(\ln x)^n\,dx = [(x^2/2)(\ln x)^n]_1^e - \int_1^e (nx/2)(\ln x)^{n-1}\,dx$ M1A1

$\Rightarrow I_n = e^2/2 - (n/2)I_{n-1} \qquad \Rightarrow I_{n+1} = \dfrac{e^2}{2} - \dfrac{(n+1)}{2}I_n$ A1

$H_k : I_k = A_k e^2 + B_k$, where A_k and B_k are rational B1

$H_k \Rightarrow I_{k+1} = e^2/2 - (k+1)(A_k e^2 + B_k)/2$,

$= A_{k+1}e^2 + B_{k+1}$, where $A_{k+1} = 1/2 - (k+1)A_k/2$, $B_{k+1} = -(k+1)B_k/2$, M1

$\Rightarrow A_{k+1}$ and B_{k+1} are rational A1

$I_1 = e^2/4 + 1/4 \Rightarrow A_1 = 1/4$, $B_1 = 1/4 \Rightarrow H_1$ is true M1A1

Completion of induction argument A1

[6]

6 Obtains an equation in y not involving radicals, e.g.,

$y(y+1)^2 = 1$ M1

$\Rightarrow \ldots \Rightarrow y^3 + 2y^2 + y - 1 = 0$ (AG) A1

[2]

(i) $S_2 = -2$ B1

$S_4 = 4 - 2 = 2$ M1A1

[3]

(ii) $S_6 = -2S_4 - S_2 + 3 = 1$ M1A1

OR

$\Sigma\alpha^2 = -2$, $\Sigma\alpha^2\beta^2 = 1$, $\alpha^2\beta\gamma^2 = 1$

$S_6 = (\Sigma\alpha^2)^3 - 3\Sigma\alpha^2\Sigma\alpha^2\beta^2 + 3\alpha^2\beta^2\gamma^2$ M1

$= (-2)^3 - 3 \times (-2) \times 1 + 3$

$= -8 + 6 + 3$

$= 1$ A1

$S_8 = -2S_6 - S_4 + S_2 = -6$ M1A1

[4]

7 **(i)** Solves any 2 of the equations:

$4 + 2\lambda = 4 + \mu$, $-2 + \lambda = -5 - \mu$, $-4\lambda = 2 - \mu$

to obtain $\lambda = -1$, $\mu = -2$ — M1A1

Checks consistency with the third equation — A1

[3]

(ii) $P = |(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}).(5\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) / \sqrt{38}|$ — M1A1

$= 7 / \sqrt{38} = 1.14$ — A1

[3]

OR

$\mathbf{n} = -5\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$ — M1

Plane is $5x + 2y + 3z = 16$ — A1

$$P = \frac{15 - 10 + 18 - 16}{\sqrt{5^2 + 2^2 + 3^2}} = \frac{7}{\sqrt{38}}$$ — A1

OR

Plane is $5x + 2y + 3z = 16$ (as above) — M1A1

Sub. general pt on perpendicular $\begin{pmatrix} 3 + 5t \\ -5 + 2t \\ 6 + 3t \end{pmatrix} \Rightarrow t = -\frac{7}{38}$

$$\Rightarrow P = \left| \begin{pmatrix} 5t \\ 2t \\ 3t \end{pmatrix} \right| = 1.14$$ — A1

(iii) $(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) \times (2\mathbf{i} + \mathbf{j} - 4\mathbf{k}) = 6\mathbf{i} + 8\mathbf{j} + 5\mathbf{k}$

OR $(\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}) \times (2\mathbf{i} + \mathbf{j} - 4\mathbf{k}) = -6\mathbf{i} - 8\mathbf{j} - 5\mathbf{k}$, etc. — B1

$d = |6\mathbf{i} + 8\mathbf{j} + 5\mathbf{k}| / \sqrt{21} = \sqrt{125/21} = 2.44$ — M1A1A1

[4]

OR

Let Q be the foot of the perpendicular from P to l, and A be the known point on l_1

$$AQ = \left| (\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}).\frac{(2\mathbf{i} + \mathbf{j} - 4\mathbf{k})}{\sqrt{21}} \right| = \frac{8}{\sqrt{21}}$$ — M1A1

$AP^2 = 1^2 + (-2)^2 + 2^2 = 9$ — B1

$$PQ^2 = 9 - \frac{64}{21} = \frac{125}{21} \Rightarrow PQ = \frac{5\sqrt{5}}{21}$$ — A1

OR

$$\overrightarrow{PQ} = \begin{pmatrix} 4 + 2t \\ -2 + t \\ -4t \end{pmatrix} - \begin{pmatrix} 3 \\ -5 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 + 2t \\ 3 + t \\ -6 - 4t \end{pmatrix} \therefore \begin{pmatrix} 1 + 2t \\ 3 + t \\ -6 - 4t \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix} = 0$$ — M1

$$\Rightarrow t = -\frac{29}{21}$$ — A1

$$\overrightarrow{PQ} = \frac{1}{21}\begin{pmatrix} -37 \\ 34 \\ -10 \end{pmatrix} \Rightarrow \left|\overrightarrow{PQ}\right| = \frac{1}{21}\sqrt{37^2 + 34^2 + 10^2} = 2.44$$ — M1A1

8	$\begin{pmatrix} 4 & 1 & -1 \\ -4 & -1 & 4 \\ 0 & -1 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}\begin{pmatrix} 3 \\ -6 \\ -3 \end{pmatrix}$	M1
	$\Rightarrow$ eigenvalue = 3	A1 [2]
	Eigenvector corresponding to 4 is $\begin{pmatrix} 1 \\ -4 \\ -4 \end{pmatrix}$	M1A1 [2]
	D = diag(1 243 1024)	B1
	$\mathbf{P} = \begin{pmatrix} 1 & 1 & 1 \\ -4 & -2 & -4 \\ -1 & -1 & -4 \end{pmatrix}$ (f.t.)	A1
	$\mathbf{Q} = \mathbf{P}^{-1}$	B1
	$= \begin{pmatrix} -2/3 & -1/2 & 1/3 \\ 2 & 1/2 & 0 \\ -1/3 & 0 & -1/3 \end{pmatrix}$ ft on **P**	M1A2√
	M1 for any valid method: A2 if completely correct A1 if exactly 1 error: A0 if > 1 errors	[6]
9	**(i)** exp $(2\pi ki/5)$, k = 0, 1, 2, 3, 4 (AEF)	M1A1 [2]
	M1 for 1 correct fifth root of unity A1 for exactly 5 distinct, correct roots	
	(ii) $z^5 = 32 \exp(-2\pi i/3)$	M1
	$z_k = 2 \exp(-2\pi i/15 + 2\pi ki/5)$	A1
	Roots equally spaced on circle $\|z\| = 2$; correctly placed	M1A1 [4]
	(iii) $\sum_{k=0}^{4}(w/2)^k = \dfrac{[1-(w/2)^5]}{[1-w/2]}$	M1
	$= \dfrac{1-(1/32)(-16-16\sqrt{3}i)}{[1-w/2]}$	A1
	$= ... = (3 + \sqrt{3}i)/(2 - w)$ (AG)	A1 [3]
	(iv) Deduces from diagram in **(ii)** that minimum of $\|2 - w\|$ occurs when $w = 2e^{-2\pi i/15}$ or $2e^{\frac{28\pi}{15}i}$	M1A1 [2]
	OR Evaluates 5 possible values of $\|2 - w\|$	M1
	Identifies minimum of $\|2 - w\|$ correctly	A1

10 $\mathbf{A} = \begin{pmatrix} 1 & 4 & 12 \\ 2 & a & 12 \\ 3 & 12 & 2a \end{pmatrix} \to \ldots \to = \begin{pmatrix} 1 & 4 & 12 \\ 0 & a-8 & -12 \\ 0 & 0 & 2a-36 \end{pmatrix}$ M1A1

$\Rightarrow r(\mathbf{A}) = 3$ provided $a \neq 18$ and $a \neq 8$ A1

$\Rightarrow$ unique solution for all values of a except $a = 18$ and $a = 8$ A1

[4]

OR for first 3 marks:

det $\mathbf{A} = 0 \Rightarrow \ldots \Rightarrow a^2 - 26a + 144 = 0$ (M1A1)

$\Rightarrow a = 8$ or 18 (A1)

$a = 18 \Rightarrow 0z = -5$ which is impossible for any finite z, or equivalent contradiction M1A1

[2]

When $a = 8$ system reduces to 2 equations:

$x + 4y = 2$ and $z = 1/4$ M1

All solutions then of form:

$x = \lambda$, $y = (2 - \lambda)/4$, $z = 1/4$ where λ is real A1

May be parametrised in any equivalent way

$\Rightarrow$ an infinite number of solutions A1

$\lambda + (2 - \lambda)/4 + 1/4 = 1$ M1

$\Rightarrow \lambda = 1/3 \Rightarrow x = 1/3$, $y = 5/12$, $z = 1/4$ A1

[5]

11 EITHER

$y' = 3z^2z'$ B1

$y'' = 6z(z')^2 + 3z^2z''$ B1

Obtains given y, x DE (AG) B1

[3]

CF: $y = e^{-x}[A\cos 2x + B\sin 2x]$ M1A1

PI: $y = x$ M1A1

GS: $y = e^{-x}[A\cos 2x + B\sin 2x] + x$ A1

When $x = 0$, $y = 1$, $y' = -2$ (both) B1

Finds A, B : $A = 1$, $B = -1$ (both) M1A1

$z = [e^{-x}(\cos 2x - \sin 2x) + x]^{1/3}$ A1

[9]

OR for previous 4 marks:

$z = [e^{-x}(A\cos 2x + B\sin 2x) + x]^{1/3}$ (B1)

When $x = 0$, $z = 1$, $z' = -2/3$ (both) (B1)

Finds A, B : $A = 1$, $B = -1$ (both) (M1A1)

For large positive x : $e^{-x}(\cos 2x - \sin 2x) \approx 0$ M1

$\Rightarrow z \approx x^{1/3}$ (AG) A1

[2]

11 OR

(i)	$x = 1$	B1
	$y = 1 + O(1/x)$ as $\lvert x\rvert \to \infty$	M1
	Second asymptote is $y = 1$	A1
		[3]
(ii)	$x(x + 1) / (x - 1)^2 = 1 \Rightarrow x = 1/3,\ y = 1$	M1A1
		[2]
(iii) (a)	$dy / dx = 0 \Rightarrow [(2x + 1)(x - 1)^2 - 2x(x - 1)(x + 1)] / (x - 1)^4 = 0$	M1
	$\Rightarrow x = 1/3,\ y = -1/8$	M1A1
(b)	$dy / dx = -(3x + 1) / (x - 1)^3$	M1
	$\{x : x < -1/3\} \cup \{x : x > 1\}$	A1√A1
	ft.	[6]
(iv)	Sketch:	
	Left-hand branch with approximately correct shape and location and passing through the origin and (–1, 0).	B1
	Intersection with $y = 1$ and location of minimum point consistent with results of **(ii)** and **(iii)** (cwo)	B1
	Right-hand branch with approximately correct forms at infinity	B1
		[3]